

VaR and Expected Shortfall¹

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1 Value at Risk

1.1 The Measure

What is value at risk (VaR)? and why should we care?

Imagine you are monitoring the risk of a portfolio, a nature question to ask is “how bad can it be tomorrow?” or “How much can I loss tomorrow?” VaR gives a simple answer by providing a *number* that can summarize the total risk of your portfolio.

–For example, you can make the following statement:

I am 99% sure that we will not suffer from a loss of more than 1 million U.S. dollar in the next 5 days.

1 million is the VaR number. It relates to two things: (1) the confidence level: e.g. 99%; (2) time horizon: e.g. 5 days.

–If you change the confidence level, VaR changes correspondingly. For example:

I am 99.9% sure that we will not suffer from a loss of more than 2 million U.S. dollar in the next 5 days.

Approximation: Time horizon conditional on a fixed confidence level

N -day $x\%$ VaR = \sqrt{N} * (1-day $x\%$ VaR)

For example: 5-day 99% VaR = $\sqrt{5}$ * (1-day 99% VaR). Note that, it is an approximation.

Another example: if regulator requires a bank to hold capital equals 3 times the 5-day 99% VaR. Then, the bank has to hold, *approximately*, $3 * (\sqrt{5}) * (1\text{-day } 99\% \text{ VaR})$

1.2 Historical Simulation

Illustration: Historical simulation method is based on the idea of using the past to tell the future. The assumption is that history repeats itself.

What do we want to know? Let’s say I have \$100 invested in a stock today. I want to be 99% sure that I will not suffer from a loss of more than x U.S. dollar tomorrow. Then, what’s the value of this x ?

What to do? First, we obtain the historical prices of this stock for the past 1,001 days (sample in daily frequency). This allows us to calculate 1,000 daily returns. Using these 1,000 daily returns plus today’s value of our investment (\$100), we can have 1,000 possible outcomes for tomorrow’s P/L. In other words, we build up 1,000 scenarios of what can happen between today and tomorrow using the past information. Second, sort the 1,000 daily returns from the lowest to the highest. The tenth ($1\% * 1,000$) lowest gain (highest loss) is the value of x if one wants to be 99% sure. Note that x takes the absolute value of the tenth lowest gain. For example, if the tenth lowest gain is -1.26. Then, x equals 1.26. The statement is that I am 99% sure that we will not suffer from a loss of more than 1.26 million U.S. dollar tomorrow (in the next 1 day).

Note: *the key assumption is that we believe the sample (1,001 historical prices) can represent what will happen between today and tomorrow.*

1.2.1 Historical Simulation: Examples

We use three examples to calculate the 1-day 99% VaR for a portfolio using historical simulation method.

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Table 1: Portfolio with Single Asset

Day	Price(\$)	Return	Investment	Scenarios	Tomorrow's P/L(\$)	Scenarios	Sorting Tomorrow's P/L(\$)
0	201.82135		Asset				
1	197.66101	-2.06%	\$400	1	-8.245587496	19	-8.65568581
2	198.86664	0.61%		2	2.439787177	20	-8.463237473
3	200.83316	0.99%		3	3.955458834	1	-8.245587496
4	199.37062	-0.73%		4	-2.912943261	10	-5.971029613
5	200.90234	0.77%		5	3.073116776	14	-3.162885968
6	202.78981	0.94%		6	3.75797706	4	-2.912943261
7	202.08818	-0.35%		7	-1.383953168	8	-2.249404185
8	200.95174	-0.56%		8	-2.249404185	17	-1.71690854
9	203.23450	1.14%		9	4.543899039	7	-1.383953168
10	200.20070	-1.49%		10	-5.971029613	13	-0.325557621
11	204.77611	2.29%		11	9.141644406	15	1.391157103
12	206.37698	0.78%		12	3.127075743	2	2.439787177
13	206.20902	-0.08%		13	-0.325557621	5	3.073116776
14	204.57848	-0.79%		14	-3.162885968	12	3.127075743
15	205.28998	0.35%		15	1.391157103	16	3.735397156
16	207.20708	0.93%		16	3.735397156	6	3.75797706
17	206.31769	-0.43%		17	-1.71690854	3	3.955458834
18	210.52747	2.04%		18	8.161739385	9	4.543899039
19	205.97182	-2.16%		19	-8.65568581	18	8.161739385
20	201.61385	-2.12%		20	-8.463237473	11	9.141644406

In the first example, the portfolio consists of a single asset. In the second example, the portfolio contains two assets denominated in the same currency. In the third example, the portfolio contains three assets denominated in three different currencies.

The VaR is calculated at the end of today. The assumption is that the portfolio will remain unchanged over the next business day.

– **Example 2.1:**

Setting: (1) Calculating 1-day 90% VaR using the 21 historical daily prices; (2) Portfolio contains a single asset: \$400 invested in stock A (Asset 1), which price is denominated in U.S. Dollar.

The detailed calculation is shown in Table 1.

As shown in Table 1, 21 historical prices gives 20 daily returns. Those 20 daily returns provide 20 scenarios of what can happen between today and tomorrow. With the assumption that the value of investment is \$400, we can have 20 possible outcomes for tomorrow's profit and loss. Ranking these 20 P/Ls from lowest to highest, the second (10%*20) highest loss (the 20th scenario as shown in the table) is the one that we are looking for.

Now, we can make the following statement: I am 90% sure that we will not suffer from a loss of more than around 8.46 U.S. dollar tomorrow/the next day.

– **Example 2.2:**

Setting: (1) Calculating 1-day 90% VaR using the 21 historical daily prices; (2) Portfolio contains two assets: \$400 invested in stock A (Asset 1) and \$200 invested in stock B (Asset 2). Both prices are denominated in U.S. Dollar.

The detailed calculation is shown in Table 2.

As shown in Table 2, 21 historical prices gives 20 daily returns. Those 20 daily returns provide 20 scenarios of what can happen between today and tomorrow. With the assumption that the value of my investment is \$400 in Asset 1 and \$200 in Asset 2, we can have 20 possible outcomes for tomorrow's *total*³ profit and loss. Ranking

³The calculation process is similar to what we have done in Example 1 (shown in Table 1). We perform the same calculation for both assets separately. The "P/L of the portfolio" is the sum of "P/L of Asset 1" and "P/L of Asset 2". Then, we sort the P/Ls of the portfolio to find out the loss value.

Table 2: Portfolio with Two Assets both denominated in U.S. Dollar

Day	Asset1	Asset2	Asset1	Asset2	Investments		Scenarios	Tomorrow's P/L(\$)		
	Price(\$)	Price(\$)	Return	Return	Asset1	Asset2		Asset1	Asset2	Portfolio
0	201	185			\$400	\$200				
1	197	182	-1.99%	-1.62%			1	-7.96	-3.24	-11.20
2	198	188	0.51%	3.30%			2	2.03	6.59	8.62
3	200	189	1.01%	0.53%			3	4.04	1.06	5.10
4	199	196	-0.50%	3.70%			4	-2.00	7.41	5.41
5	200	186	0.50%	-5.10%			5	2.01	-10.20	-8.19
6	202	187	1.00%	0.54%			6	4.00	1.08	5.08
7	202	188	0.00%	0.53%			7	0.00	1.07	1.07
8	200	193	-0.99%	2.66%			8	-3.96	5.32	1.36
9	203	194	1.50%	0.52%			9	6.00	1.04	7.04
10	200	196	-1.48%	1.03%			10	-5.91	2.06	-3.85
11	204	195	2.00%	-0.51%			11	8.00	-1.02	6.98
12	206	200	0.98%	2.56%			12	3.92	5.13	9.05
13	206	201	0.00%	0.50%			13	0.00	1.00	1.00
14	204	197	-0.97%	-1.99%			14	-3.88	-3.98	-7.86
15	205	200	0.49%	1.52%			15	1.96	3.05	5.01
16	207	196	0.98%	-2.00%			16	3.90	-4.00	-0.10
17	206	198	-0.48%	1.02%			17	-1.93	2.04	0.11
18	210	203	1.94%	2.53%			18	7.77	5.05	12.82
19	205	204	-2.38%	0.49%			19	-9.52	0.99	-8.54
20	201	206	-1.95%	0.98%			20	-7.80	1.96	-5.84

these 20 cases from lowest to highest, the second (10%*20) highest loss (the 19th scenario) is the one that we are looking for.

Now, we can make the following statement: I am 90% sure that we will not suffer from a loss of more than 8.54 U.S. dollar tomorrow (the next day).

– **Example 2.3:**

Setting: (1) Calculating 1-day 90% VaR using the 21 historical daily prices; (2) Portfolio contains three assets: \$1,000 invested in stock A (Asset 1), £3,000 invested in stock B (Asset 2), and € 1,000 invested in stock C (Asset 3). The prices of Assets 1, 2, and 3 are denominated in U.S. Dollar, GBP, and EUR.

The detailed calculation is shown in Table 3.

As shown in Table 3, 21 historical prices gives 20 daily returns. These 20 daily returns provide 20 scenarios of what can happen between today and tomorrow. The twist here is to first transform all assets' prices into local currency. In our case, it's the U.S. Dollar⁴. Then, we can calculate the dollar returns for each asset. With the assumption that the value of investment is \$1,000 in Asset 1, £3,000 in Asset 2 and € 1,000 in Asset 3, we can have 20 possible outcomes for tomorrow's profit and loss of each asset. Tomorrow's *total* profit and loss of the portfolio is obtained by summing up all three assets' P/Ls. This gives us 20 possible outcomes for the P/L of our portfolio tomorrow. Ranking these 20 P/Ls from lowest to highest, the second (10%*20) highest loss (the 3rd scenario) is the one that we are looking for.

Now, we can make the following statement: I am 90% sure that we will not suffer from a loss of more than 48.3 U.S. dollar tomorrow (the next day).

1.3 Variance-Covariance Approach

When using the variance-covariance approach to calculate the VaR, one of the most important steps is to obtain the volatility of the portfolio, that is the variance-covariance matrix.

Here are the steps: (1) get the historical prices of the underlying assets (for a given period); (2) calculate the returns for each asset; (3) obtain the variance-covariance matrix based on those returns; (4) get the matrix

⁴If you are with a Swiss bank, all prices should be transformed into CHF.

Table 3: Portfolio with Three Assets Denominated in Different Currencies

Day	Asset1	Asset2	Asset3	Exchange Rate		Adjusted Prices(\$)		Asset1(\$)	Asset2(\$)	Asset3(\$)
	Price(\$)	Price(GBP)	Price(EUR)	USD/GBP	USD/EUR	Asset2(\$)	Asset3(\$)	Return	Return	Return
0	3232	6239	12608	1.909	1.286	11910.2	9804.0			
1	3207	6233	12260	1.907	1.283	11886.3	9555.7	-0.77%	-0.20%	-2.53%
2	3190	6231	12310	1.908	1.288	11888.7	9557.4	-0.53%	0.02%	0.02%
3	3102	6239	12232	1.891	1.277	11797.9	9578.7	-2.76%	-0.76%	0.22%
4	3121	6218	12089	1.897	1.276	11795.5	9474.1	0.61%	-0.02%	-1.09%
5	3100	6213	12177	1.892	1.274	11754.9	9558.0	-0.67%	-0.34%	0.89%
6	3124	6217	12093	1.895	1.278	11781.2	9462.4	0.77%	0.22%	-1.00%
7	3113	6212	12523	1.910	1.285	11864.9	9745.5	-0.35%	0.71%	2.99%
8	3115	6235	12262	1.893	1.286	11802.8	9534.9	0.06%	-0.52%	-2.16%
9	3097	6229	12330	1.879	1.282	11704.2	9617.7	-0.58%	-0.84%	0.87%
10	3117	6240	12281	1.897	1.291	11837.2	9512.7	0.65%	1.14%	-1.09%
11	3131	6218	12382	1.888	1.281	11739.5	9665.8	0.45%	-0.83%	1.61%
12	3150	6234	12315	1.893	1.279	11800.9	9628.6	0.61%	0.52%	-0.39%
13	3123	6231	11911	1.891	1.279	11782.8	9312.7	-0.86%	-0.15%	-3.28%
14	3089	6234	11949	1.886	1.274	11757.3	9379.1	-1.09%	-0.22%	0.71%
15	3053	6228	11970	1.892	1.275	11783.3	9388.2	-1.17%	0.22%	0.10%
16	3100	6234	12530	1.905	1.283	11875.7	9766.1	1.54%	0.78%	4.03%
17	3115	6238	12617	1.901	1.279	11858.4	9864.7	0.48%	-0.15%	1.01%
18	3122	6241	12819	1.903	1.280	11876.6	10014.8	0.22%	0.15%	1.52%
19	3112	6233	12430	1.892	1.280	11792.8	9710.9	-0.32%	-0.71%	-3.03%
20	3055	6235	12487	1.880	1.278	11721.8	9770.7	-1.83%	-0.60%	0.62%

Continue

Asset1(\$) Return	Asset2(\$) Return	Asset3(\$) Return	Investment			Scenarios	Tomorrow's P/L (\$)			
			Asset1 \$1000	Asset2 £3000	Asset3 €1000		Asset1	Asset2	Asset3	Portfolio
-0.77%	-0.20%	-2.53%				1	-7.7	-6.0	-25.3	-39.1
-0.53%	0.02%	0.02%				2	-5.3	0.6	0.2	-4.5
-2.76%	-0.76%	0.22%				3	-27.6	-22.9	2.2	-48.3
0.61%	-0.02%	-1.09%				4	6.1	-0.6	-10.9	-5.4
-0.67%	-0.34%	0.89%				5	-6.7	-10.3	8.9	-8.2
0.77%	0.22%	-1.00%				6	7.7	6.7	-10.0	4.4
-0.35%	0.71%	2.99%				7	-3.5	21.3	29.9	47.7
0.06%	-0.52%	-2.16%				8	0.6	-15.7	-21.6	-36.7
-0.58%	-0.84%	0.87%				9	-5.8	-25.1	8.7	-22.1
0.65%	1.14%	-1.09%				10	6.5	34.1	-10.9	29.6
0.45%	-0.83%	1.61%				11	4.5	-24.8	16.1	-4.2
0.61%	0.52%	-0.39%				12	6.1	15.7	-3.9	17.9
-0.86%	-0.15%	-3.28%				13	-8.6	-4.6	-32.8	-46.0
-1.09%	-0.22%	0.71%				14	-10.9	-6.5	7.1	-10.3
-1.17%	0.22%	0.10%				15	-11.7	6.6	1.0	-4.0
1.54%	0.78%	4.03%				16	15.4	23.5	40.3	79.2
0.48%	-0.15%	1.01%				17	4.8	-4.4	10.1	10.6
0.22%	0.15%	1.52%				18	2.2	4.6	15.2	22.1
-0.32%	-0.71%	-3.03%				19	-3.2	-21.2	-30.3	-54.7
-1.83%	-0.60%	0.62%				20	-18.3	-18.1	6.2	-30.2

in dollar term (this is the same for all approaches. We start with percentage returns, then, get the single or matrix % in dollar term; (5) The 99% Value-at-Risk = 2.33*stdev; 95% Value-at-Risk =1.65*stdev.

– **Example 3.1:** We will use the same example⁵ in Hull (2017) as our first example. Then, we will replicate the results using Python, Matlab, and R.

Setting: (1) Position: \$10 million in Microsoft shares and \$5 million in AT&T shares; (2) Daily volatility of ‘Microsoft returns’ is 2% and daily volatility of ‘AT&T returns’ is 1%; (3) The correlation of the returns on the two shares is 0.3

Assumption: The returns of the shares have a normal distribution.

Question: What is the 1-Day 99% VaR and 10-Day 99% VaR of this portfolio?

We know 10-Day 99% VaR= $\sqrt{10}$ *1-Day 99% VaR, so we need to get 1-Day 99% VaR first. Also, by using the formula: 99% Value-at-Risk = 2.33*stdev, once we have stdev (portfolio’s dollar volatility), we can obtain the 1-Day 99% VaR. The portfolio’s dollar volatility can be calculated using each stock’s dollar volatility.

Solve:

– We are given the volatility of returns in *percentage term*. First, we change the given volatility into *dollar term*.

⁵The example here refers to the example in Two-Asset case of model-building approach introduced in Section 21.3.

$$\begin{aligned}\sigma_{Microsoft} &= 2\% * 10 = 0.2 \\ \sigma_{AT\&T} &= 1\% * 5 = 0.05\end{aligned}$$

– ‘Variance’ of the ‘change in the value of the portfolio’ during ‘1-day period’ is:

$$\begin{aligned}\sigma^2_{portfolio} &= \sigma^2_{Microsoft} + \sigma^2_{AT\&T} + 2\rho\sigma_{Microsoft}\sigma_{AT\&T} \\ &= 0.2^2 + 0.05^2 + 2 * 0.3 * 0.2 * 0.05 = 0.0485\end{aligned}$$

This means, $\sigma_{portfolio} = 0.220227$

Then, 1-Day 99% VaR = $2.33 * 0.220227 = 0.513129$ million

10-Day 99% VaR = $\sqrt{10} * 1\text{-Day 99\% VaR} = 1.622657$ million

Python: Example 3.1

```
import numpy as np # If you do not have numpy installed, type: 'python -m pip install -U pip' in
'cmd'; then, type 'pip install numpy'.)
investment=np.array([10,5])
cov_portfolio=np.array([[4e-04,6e-05],[6e-05,1e-04]])
var_portfolio=investment.T.dot(cov_portfolio).dot(investment)
stdev_portfolio=np.sqrt(var_portfolio)
var=2.33*stdev_portfolio
```

Matlab: Example 3.1

```
investment=[10,5];
cov_portfolio=[0.0004,0.00006;0.00006,0.0001];
var_portfolio=investment*cov_portfolio*transpose(investment);
stdev_portfolio=sqrt(var_portfolio);
var=2.33*stdev_portfolio;
```

R: Example 3.1

```
investment=rbind(c(10,5))
cov_matrix=rbind(c(4e-04,6e-05),c(6e-05,1e-04))
var_portfolio=investment%*%cov_matrix%*%t(investment)
stdev_portfolio=sqrt(var_portfolio)
var=2.33*stdev_portfolio
```

– **Example 3.2:** We will use the same example in Ken Abbott’s VAR lecture note as our second example. Then, we will replicate the results using Python, Matlab, and R.

Setting: (1) Position: \$100 in CAD/USD, \$50 in CHF/USD, and \$25 in DEM/USD; (2) Variance and covariance matrix is as

$$\begin{array}{ccc} 0.000037 & -0.000018 & -0.000017 \\ -0.000018 & 0.000321 & 0.000257 \\ -0.000017 & 0.000257 & 0.000227 \end{array}$$

Assumption: Return follows a normal distribution.

Question: What is the 1-Day 99% VaR of this portfolio?

– We are given the volatility of returns in *percentage term*. First, we change the given volatility into *dollar term*.

$$\begin{aligned}\sigma_{CAD/USD} &= \sqrt{0.000037} * 100 = 0.608276 \\ \sigma_{CHF/USD} &= \sqrt{0.000321} * 50 = 0.895823 \\ \sigma_{DEM/USD} &= \sqrt{0.000227} * 25 = 0.376662 \\ \rho_{CAD/USD,CHF/USD}\sigma_{CAD/USD}\sigma_{CHF/USD} &= 100 * 50 * -0.000018 = -0.09 \\ \rho_{CAD/USD,DEM/USD}\sigma_{CAD/USD}\sigma_{DEM/USD} &= 100 * 25 * -0.000017 = -0.0425 \\ \rho_{CHF/USD,DEM/USD}\sigma_{CHF/USD}\sigma_{DEM/USD} &= 50 * 25 * 0.000257 = 0.321250\end{aligned}$$

Python: Example 3.2

```
import numpy as np
investment=np.array([100,50,25])
cov_matrix=np.array([[0.000037,-0.000018,-0.000017],[-0.000018,0.000321,0.000257],
[-0.000017,0.000257,0.000227]])
var_portfolio=investment.T.dot(cov_matrix).dot(investment)
stdev_portfolio=np.sqrt(var_portfolio)
var=2.33*stdev_portfolio
```

Matlab: Example 3.2

```
investment=[100,50,25];
cov_matrix=[0.000037,-0.000018,-0.000017;-0.000018,0.000321,0.000257;-0.000017,0.000257,0.000227];
var_portfolio=investment*cov_matrix*transpose(investment);
stdev_portfolio=np.sqrt(var_portfolio);
var=2.33*stdev_portfolio;
```

R: Example 3.2

```
investment=rbind(c(100,50,25));
cov_matrix=rbind(c(0.000037,-0.000018,-0.000017),c(-0.000018,0.000321,0.000257),c(-
0.000017,0.000257,0.000227));
var_portfolio=investment%%cov_matrix%%t(investment)
stdev_portfolio=sqrt(var_portfolio)
var=2.33*stdev_portfolio
```

Python: Example 3.3

```
import numpy as np
import pandas as pd
import pandas.datareader as pdr
ticker=['AAPL','DIS','EBAY','FB']
# Position setting:
investment=np.array([0.35, 0.3, 0.25, 0.1]) # Set this directly
# Alternatively: try the belowing three lines
total=1 # Total investment as 1 million
weight=np.array([0.35,0.3,0.25,0.1]) # We set the weight
investment=total*weight # We have the same position.
data=pdr.data.get_data_yahoo(ticker, start="2018-01-01", end="2020-07-18")['Adj Close']
returns=data.pct_change()# percentage returns
cov_matrix=returns.cov()
var_portfolio=investment.T.dot(cov_matrix).dot(investment)# var-cov matrix in dollar term
stdev_portfolio=np.sqrt(var_portfolio) #volatility in dollar term
var=2.33*stdev_portfolio
# An alternative way for the last line
mean_portfolio=returns.mean().dot(weight)
mean_investment=(1+mean_portfolio)*total
from scipy.stats import norm
cutoff=norm.ppf(0.01,mean_investment,stdev_portfolio)# normality assumption kicks in
var=total-cutoff
```

–‘Variance’ of the ‘change in the value of the portfolio’ during ‘1-day period’ is:

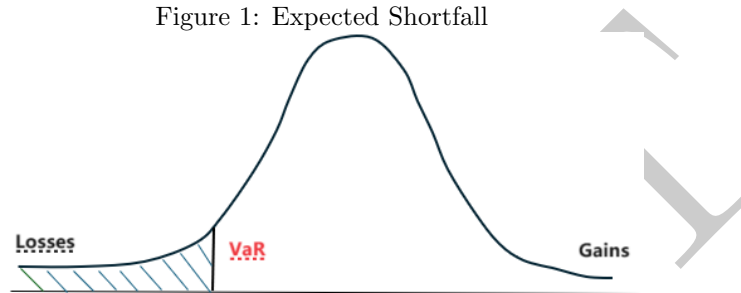
$$\sigma^2_{portfolio} = 0.608276^2 + 0.895823^2 + 0.376662^2 + 2*(-0.09) + 2*(-0.0425) + 2*(0.321250) = 1.692$$

This means, $\sigma_{portfolio} = 1.3007$

Then, 1-Day 99% VaR = $2.33 * 0.220227 = 3.03$

– **Example 3.3:** The variance and covariance matrix of the asset returns is the same as before (previous example). Now we show the entire process in Python starting from downloading the asset prices, calculating corresponding returns and the variance and covariance matrix. Then, obtaining the 1-Day 99% VaR of the portfolio. The portfolio we used here contains four assets: AAPL; DIS; EBAY; and FB. The stock prices start from Jan 01 2018 till July 18 2020. Yahoo finance is the data source.

Figure 1: Expected Shortfall



2 Expected Shortfall

Expected Shortfall (ES) is a derivative concept of value at risk, where the VaR value serves as a threshold. ES is calculated as the average losses exceeded the VaR level. It is also known as the conditional value at risk (CVaR) since it's conditional on the VaR value.

Both VaR and ES are risk measures, so why we need to introduce ES when we already have VaR? The reason is that ES gives us a more complete picture of the losses (shape of the tail). In other words, expected shortfall is a more robust risk measure comparing to the VaR.

The underlying loss distribution is important for both VaR and ES measures. Nowadays, most programs can directly calculate VaR and ES values, but the usefulness of the results depends on what kind of model we use to fit the sample. For example, if the sample shows a heavy tail feature, but we choose to fit it using a normal distribution, or the returns are asymmetric, but we choose Student's t distribution to fit it. The VaR and ES values in both cases are not reasonable and have no real implication. In addition, it is important to keep in mind that if we choose to use the historical data to predict future returns. Then, sample selection becomes extremely important, especially in times of crisis. For example, if we choose a normal period of time as the sample period, then we are underestimating the risk during the crisis. That is, the values of VaR and ES are too small. However, if we choose the historical sample from the previous crisis period instead, we will encounter another problem, that is the sample size. In this case, the VaR and ES values might be imprecise as well.

References

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