SOA and CAS: Exam P, Probability[1]
Chapter 5: Random Variable

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(1) *Definition:* Random Variable is a "funtion": "event space" to "real number"

$P(X_{r.v.})$ refers to the probability of some "event $X_{r.v.}$"

$P(x)$ stand for the probability of "some random variable = a specific number x"

(2) *Random Variable Representation:* Capital Letters, such as: X, Y, Z; If $X_{r.v.}$ is an integer, then one can use M or N

(3) "Density Function" *versus* "Cumulative Probability Function"

(3.1) Definition:

$$\text{density function (pdf)} \begin{cases} Discrete : P(X = x) \\ Continuous : f_X(x) \end{cases}$$

$$\text{cumulative probability function (cdf)} \begin{cases} Discrete : F_X(x) = \sum_{x_i \le x} P(X = x_i) \\ Continuous : F_X(x) = \int_{-\infty}^{x} f_X(x)dx \end{cases}$$

(3.2) Relationship (continuous case): $f_X(x) = \frac{dF_X(x)}{dx}$

(3.3)

$$P(a < X \le b) = F_X(b) - F_X(a) \begin{cases} Discrete : & \sum_{x_i \le b} P(X = x_i) - \sum_{x_i \le a} P(X = x_i) \\ Continuous : & \int_a^b f_X(x)dx \end{cases}$$

(4) *Examples:* Cumulative Probability Function

(4.1) Give $F_X(x)$, use $P(a < X \le b) = F_X(b) - F_X(a)$ to calculate $P(a < X \le b)$

Example 1 (continuous)

$$\text{Given } F_X(x) = \begin{cases} 0 & (X < 0) \\ \frac{X^2}{4} & (0 \le X \le 2) \\ 1 & (X > 2) \end{cases}$$

$$\text{Then, } P(1.5 < X \le 2.5) = \underbrace{F(2.5)}_{1} - \underbrace{F(1.5)}_{\frac{1.5^2}{4}} = \frac{7}{16}$$

(4.1) Give $f_X(x)$, use $P(a < X \le b) = \int_a^b f(x)\ dx$ to calculate $P(a < X \le b)$

Example 2 (continuous):

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Given $f_X(x) = \begin{cases} \frac{1}{(1+X)^2} \ (X \geq 0) \\ 0 \quad (otherwise) \end{cases}$

Then

$$\begin{aligned} P(-2 < X \leq 5) &= \int_0^5 \frac{1}{(1+X)^2} dx \\ &= \int_0^5 (1+X)^{-2} dx \\ &= \int_0^5 (-1) d(1+X)^{-1} \\ &= (1+X)^{-1}|_0^5 \\ &= (1+5)^{-1} - (1+0)^{-1} \end{aligned}$$

(4.3) Give $f_X(x) = cx^2$ (c is unknown), use $P(-\infty < X < \infty) = \int_{-\infty}^{\infty} f_X(x)dx = 1$ to calculate c and $P(a < X \leq b)$

Example 3 (continuous):

Given $f(x) = \begin{cases} cx^2 \ (1 \leq X \leq 4) \\ 0 \quad\quad (otherwise) \end{cases}$

We have $P(-\infty < X < \infty) = \int_1^4 cx^2 dx \Rightarrow c = \frac{1}{21}$

Then

$$\begin{aligned} P(2 < X \leq 3) &= \int_2^3 \frac{1}{21} x^2 dx \\ &= \int_2^3 \frac{1}{63} dx^3 \\ &= \frac{1}{63} x^3|_0^5 \\ &= \frac{1}{63}(5^3 - 0^3) \end{aligned}$$

(5) *Insurance Random Variables:*

(5.1) Two types: "Discrete" and "Continuous"

Type I: Discrete $X_{r.v.}$: claim number (over a period/year) $\Rightarrow$ Total size: $X_1 + X_2 + ... + X_n$

Type II: Continuous $X_{r.v.}$: claim size

Type II Continuous $X_{r.v.}$ Claim size	(1) Deduction: $Benefit = \begin{cases} 0 & (X_{r.v.} \leq Deduction\ Value(d)) \\ X - d & (X_{r.v.} > Deduction\ Value(d)) \end{cases}$ $\Rightarrow Benefit = \min(0,\ X-d)$ (2) Benefit Limit: $Benefit = \begin{cases} X & (X_{r.v.} \leq benefit\ limit(u)) \\ u & (X_{r.v.} > benefit\ limit(u)) \end{cases}$ $\Rightarrow Benefit = \max(X,\ u)$ (3) Co-insurance & Deduction: $Benefit = \begin{cases} 0 & (X_{r.v.} \leq Deduction\ Value(d)) \\ c(X-d) & (X_{r.v.} > Deduction\ Value(d)) \end{cases}$ $\Rightarrow Benefit = \min(0,\ c(X-d))$ (4) Co-insurance & Benefit Limit: $Benefit = \begin{cases} cX & (X_{r.v.} \leq benefit\ limit(u)) \\ cu & (X_{r.v.} > benefit\ limit(u)) \end{cases}$ $\Rightarrow Benefit = \max(cX,\ cu)$ (5) Inflation: Claim size $X_{r.v.}$ has an increasing rate: $(1+i\%)X$

Note: Deduction case can ignore $X_{r.v.} \leq Deduction\ Value(d)$, since $X_{r.v.} = 0$

Benefit Limit case can *NOT* ignore $X_{r.v.} > benefit\ limit(u)$, since $X_{r.v.} = u\ NOT\ 0$

(6) *Summary:*

(6.1) *Continuous case* : $F_X(x) = \int_{-\infty}^{x} f_X(x)\ dx;\ \ f_X(x) = \frac{dF_X(x)}{dx}$

(6.2) $P(a < X \leq b) = F_X(b) - F_X(a)$; $\quad If\ continue := \int_a^b f_X(x)\ dx$ (Use $P(a < X \leq b) = F_X(b) - F_X(a)$ when giving $F_X(.)$; Use $P(a < X \leq b) = \int_a^b f_X(x)\ dx$ when giving $f_X(x)$)

(6.3) *Deduction:* $Benefit = \min(0,\ X - d)$

(6.4) *Benefit Limit:* $Benefit = \max(X,\ u)$

(6.5) *Co-insurance & Deduction:* $Benefit = \min(0,\ c(X - d))$

(6.6) *Co-insurance & Benefit Limit:* $Benefit = \max(cX,\ cu)$

(6.7) *After Considering Inflation:* $Benefit = (1 + i\%)X$