SOA and CAS: Exam P, Probability[1]
Chapter 24: Order Statistics

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(1) *Maximum*

Condition 1: Y is the maximum of a series of X_i

Condition 2: X_i and X_j are independent

If both conditions are met, then

$$F_Y(x) = F_{X_1}(x) * F_{X_2}(x) \cdots * F_{X_N}(x) \tag{1}$$

(Question: Type A) Calculate the *probability* of the max random variable:

Example A.1:

A sample consists of X_1, X_2, X_3;
$X_1 \sim Uniform[0,1]$, $X_2 \sim Uniform[0,1]$, $X_3 \sim Uniform[0,1]$;
$Y = max(X_1, X_2, X_3)$;
Question: what is P(Y>0.7)?

Recall eq(1), we have $F_Y(x) = F_{X_1}(x) * F_{X_2}(x) * F_{X_3}(x)$

That is $P(Y \leq 0.7) = \frac{0.7-0}{1-0} * \frac{0.7-0}{1-0} * \frac{0.7-0}{1-0} = (\frac{0.7-0}{1-0})^3$, which gives $P(Y > 0.7) = 1 - P(Y \leq 0.7)$

Example A.2:

A sample consists of X_1, X_2, X_3
X_i: $f(x) = \frac{10}{x^2}$ $(X > 10)$
$Y = max(X_1, X_2, X_3)$
Question: what is P(Y≤25))?

From eq(1), we know that $P(Y \leq 25) = [P(X \leq 25)]^3$, where $P(X \leq 25) = \int_{10}^{25} f(x)\, dx = \int_{10}^{25} \frac{10}{x^2}\, dx$

(Question: Type B) Calculate the *Expectation* of the max random variable:

Step 1: get $F_X(x)$
Step 2: get $F_Y(x)$
Step 3: get $f_Y(x)$
Step 4: get $E(Y)$

Example B.1:

A sample consists of 4 claims, each claim $X_i \sim f(x) = \frac{2}{x^3}$ $(x > 1)$
$Y = max(X_1, X_2, X_3, X_4)$
Question: what is E(Y)?

Step 1: get $F_X(x) = \int_1^x \frac{2}{x^2} dx = 1 - \frac{1}{x^2}$ $(x > 1)$
Step 2: get $F_Y(x) = (1 - \frac{1}{x^2})^4$
Step 3: get $f_Y(x) = 4 * (1 - \frac{1}{x^2})^3 * \frac{2}{x^2} = 8 * (\frac{1}{x^3} - \frac{3}{x^5} + \frac{3}{x^7} - \frac{1}{x^9})$
Step 4: get $E(Y) = \int_1^\infty x * 8 * (\frac{1}{x^3} - \frac{3}{x^5} + \frac{3}{x^7} - \frac{1}{x^9}) dx = 3.6571$

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(2) *Minimum*

Condition 1: Y is the minimum of a series of X_i

Condition 2: X_i and X_j are independent

If both conditions are met, then $F_Y(x) = 1 - [1 - F_X(x)]^n$

$$\begin{aligned} F_Y(x) &= 1 - [1 - F_X(x)]^n \\ f_Y(x) &= n * [1 - F_X(x)]^{n-1} * f_X(x) \end{aligned} \tag{2}$$

(3) *Exponential*

Example 3.1

Y is the minimum of a series of X_i (total of n $X_i : (X_1, ...X_i, ...X_n)$)
$X_i \sim Exp(\theta)$
Question: what is the distribution of Y?

Trick: $Y = \min(X_1, ...X_i, ...X_n) \sim Exp(\frac{\theta}{n})$

Example 3.2 (K^{th} X_i)

Y: K^{th} X_i, (total of n X_i)
Question: what is $E(Y)$?

Trick: $E(Y)=\frac{k*\theta}{n+1}$,where $X_i \sim Exp(\theta)$