SOA and CAS: Exam P, Probability[1]
Chapter 22: Normal Distribution (Continuous)

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(1) *Definition*: $X \sim N(\mu, \sigma^2)$, *where* μ and σ^2 stand for the mean and variance.

	Probability Distribution Function	Mean	Variance	2^{nd} Raw Moment
Normal Distribution	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (-\infty < x < \infty)$	μ	σ	-
Standard Normal Distribution	$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (-\infty < x < \infty)$	0	1	-
Lognormal Distribution $\ln X \sim N(\mu, \sigma^2)$	$\ln X \sim N(\mu, \sigma^2)$	$e^{\mu+\frac{\sigma^2}{2}}$	-	$e^{2\mu+2\sigma^2}$

(1.1) If $X \sim N(\mu, \sigma^2)$, then $aX + b \sim Normal(a\mu + b, a^2\sigma^2)$

Proof: $E(aX + b) = aE(X) + b = a * \mu + b$

$$Var(aX + b) = a^2 Var(X) = a^2\sigma^2$$

(1.2) If X_1 and X_2 are independent, then $aX_1 + bX_2 + c \sim N(a\mu_1 + b\mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$

Proof: $E(aX_1 + bX_2 + c) = aE(X_1) + bE(X_2) + c = a\mu_1 + b\mu_2 + c$

$$Var(aX_1 + bX_2 + c) = Var(aX_1 + bX_2) \overset{independent}{=} a^2 Var(X_1) + b^2 Var(X_2)$$

$$= a^2\sigma_1^2 + b^2\sigma_2^2$$

(2) *Two Types of Exam Questions: Cumulative Probability and Quantile*

(2.a) Cumulative Probability
For example: $X \sim N(\mu = 5, \sigma^2 = 4^2)$
Question: What is $Q(X > 7)$?

Solve: $Q(X > 7) = 1 - Q(X \leq 7) = 1 - Q(\underbrace{\frac{X-5}{4}}_{Z} \leq \underbrace{\frac{7-5}{4}}_{check\ table})$

$$= 1 - Q_Z(0.5)$$

(2.b) Quantile

For example: $X \sim N(\mu = 10, \sigma^2 = 5^2)$
Question: What is the 67^{th} *percentile of* X?

Solve: 67^{th} *percentile of* $X = 67^{th}$ *percentile of* Z. After checking the table, we have $Z_P^{67th} = 0.44$
Then, use $Z_P^{67th} = \frac{X_P^{67th} - 10}{5}$ to get $X_P^{67th} = 12.2$
Recall, $X_p = \mu + \sigma Z_p$, X and Z have the same quantile

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(3) *Independent*

If (i) $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$
(ii) X_1 and X_2 are independent
Then: $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$, $\quad aX_1 + bX_2 + c \sim N(a\mu_1 + b\mu_2 + c, a^2\sigma_1^2 + b^2\sigma_2^2)$

For example: $X_1 \sim N(80, (\sqrt{30})^2)$, $X_2 \sim N(20, (\sqrt{10})^2)$, X_1 and X_2 are independent

Question: $P(X_1 + X_2 \leq 90) =?$

Solve: $X_1 + X_2 \sim N(\underbrace{80 + 20}_{\mu}, \underbrace{30 + 10}_{\sigma^2})$

$$\text{Thus, } P(X_1 + X_2 \leq 90) = P(\underbrace{\frac{(X_1 + X_2) - (80 + 20)}{\sqrt{30 + 10}}}_{Z} \leq \tfrac{90-(80+20)}{\sqrt{30+10}})$$

$$= P(Z \leq \underbrace{-1.581}_{\text{negative cannot check table}})$$

$$\overset{symmetric}{=} P(Z > 1.581)$$

$$= 1 - P(Z \leq 1.581)$$

$$= 1 - 0.9430 = 0.0570$$

$$= 0.0570$$

(4) *Log-Normal:* X is log-normal $\Longleftrightarrow lnX \sim N(\mu, \sigma^2)$

For example: Loss follows a log-normal, $lnX \sim N(\mu = 8, \sigma^2 = 2^2)$

Question: $P(loss \leq 10000) =?$

$$\text{Solve: } P(loss \leq 10000) = P(\ln(loss) \leq \ln(10000)) = P(\underbrace{\frac{\ln(loss) - \mu}{\sigma}}_{Z} \leq \underbrace{\frac{\ln(10000) - \mu}{\sigma}}_{0.605})$$

$$= \underbrace{P(Z \leq 0.605)}_{\Phi(0.605)}$$