SOA and CAS: Exam P, Probability[1]
Chapter 21: Exponential Distribution (Continuous)

Yi Li [2]

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(1) *Definition*: $f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}} (x \geq 0)$

For example: the probability of waiting time is 5 minutes: $f(x = 5) = \frac{1}{\theta}e^{-\frac{5}{\theta}}$

the probability of approval time is 3 minutes: $f(x = 3) = \frac{1}{\theta}e^{-\frac{3}{\theta}}$

Exponential distribution is a continuous distribution. Thus, the question will be about cdf. For example: the "waiting time" to be at least 5 mintues, or, less than 3 minutes.

Generally speaking:
Binominal and Poission distributions are both discrete distributions. The questions can be about pdf and cdf. Exponential and Normal distributions are both continuous distributions. Therefore, the questions will be about cdf. Note, n in Exponential distribution can only take nonnegative values ($n \geq 0$). Thus, when calculating cdf ($\int_0^x (pdf)dx$), the lower bound is 0. In terms of normal distribution, most questions require you to standardize X first.

Exponential Distribution	
Probability Distribution Function	$f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}} \quad (x \geq 0)$
Cumulative Distribution Function	$F(x) = 1 - e^{-\frac{x}{\theta}} \quad (x \geq 0)$
Mean	$E(X) = \theta$
Variance	$Var(X) = \theta^2$
Percentile	$X_P^{70\%} = -\theta * \ln(1 - 70\%)$

(2) *Memoryless*:

Give: $X \sim Exponential(\theta)$, then $X - x \mid X \geq x \sim Exponential(\theta)$.
Notice, here we have $X - x$, which refers to the "future X"

For example: the waiting time $X \sim Exponential(\theta)$ ($f(x) = \frac{1}{\theta}e^{-\frac{x}{\theta}}$)
Assume you have already waited 20 minutes, then the *remaining waiting time* still follows $Exponential(\theta)$. That is the *probability of waiting time to be smaller than 5 minutes* is $\int_0^5 \frac{1}{\theta}e^{-\frac{x}{\theta}}\ dx$. The total waiting time is $(20 + \int_0^5 \frac{1}{\theta}e^{-\frac{x}{\theta}}\ dx)$

(3) *Gamma Distribution (sum of n independent exponential* distributions *with the same θ*)**:**

$Z = X_1 + X_2 ... + X_n$ *where* $X_i \sim Exponential(\theta)$
All Xs share the same θ. X_i and X_j are mutually independent
Then, $Z \sim Gamma(n, \theta)$, *where* n is the total number of X_i

Probability Distribution Function	$f(x) = \frac{x^{n-1}e^{-\frac{x}{\theta}}}{\Gamma(n)\theta^n} \quad (x > 0)$ where $\Gamma(n) = (n-1) * (n-2) * ... * 1$

For example: A plan requires 2 approvals. The "first approval time" $X_1 \sim Exponential(3)$. Once you get the first approval, the plan will be sent to the second supervisor for "the second approval". The "second approval time" $X_2 \sim Exponential(3)$. Then, we have $X_1 + X_2 = Z \sim Gamma(n = 2, \theta = 3) \iff$ P("total approval time"≤ 5 *days*) $= P(Z \leq 5) = F(5) = \int_0^5 \frac{x^{2-1}e^{-\frac{x}{\theta}}}{\Gamma(2)\theta^2} dx$

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[2]Email: liyifinhub@outlook.com. This note was drafted when I was preparing for the exam. Please email me if you find any errors. My personal website http://www.yilifinhub.com

Alternatively, one can use the *law of total probability:* $P(A) = \sum_n P(A \cap B_n)$. Let x be "the second approval time". Then, $(5-x)$ is "the first approval time": $P(Z \leq 5) = \underbrace{\int_0^5}_{\textit{accumulate} \text{ over the } 2^{nd} \text{ approval time}} \underbrace{(1 - e^{-\frac{5-x}{3}})}_{\text{cdf of the } 1^{st} \text{ approval time}} * \underbrace{\frac{1}{3}e^{-\frac{5-x}{3}}}_{\text{the } 2^{nd} \text{ approval time}} dx$

(4) *Meaning of* θ*, where* $X \sim Exponential(\frac{1}{\theta})$

For example: An average 5 tornadoes in 4 years $\Longleftrightarrow$ it takes 0.8 years for 1 tornado (1 claim) to occur. Let X be the number of years until the next tornado. Then, $X \sim$ *Exponential* with mean $\theta = 0.8$