SOA and CAS: Exam P, Probability[1]
Chapter 20: Poisson Distribution (Discrete)

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(1) *Definition*:

$\Pr(N=k)$ stands for "within one unit time", probability of "the number of occurrence = k"

$$\begin{aligned} \Pr(N &= k) = e^{-\lambda}\frac{\lambda^n}{n!} \quad (n=0,1,2,...) \\ E(N) &= \lambda \\ Var(N) &= \lambda \\ E(N^2) &= \lambda + \lambda^2 = Var(N) + [E(N)]^2 \end{aligned}$$

For example:

(i) the probability of ("in one week", happens "$\underbrace{\textit{exactly } 10 \textit{ rainstorms}}_{n=10}$")

(ii) the probability of ("in a minute", arriving "$\underbrace{\textit{exactly } 3 \textit{ cars}}_{n=3}$")

(iii) the probability of ("in one year", happens "$\underbrace{\textit{exactly } 5 \textit{ snowtorms}}_{n=5}$")

(iv) the probability of ("in one year", arriving $\leq$ "2 *cars*") $= \Pr(N=0) + \Pr(N=1) + \Pr(N=2)$)

(2) *Independent*:

Give : N_1, N_2, and N_3 are mutually indepedent
$N_1 \sim Poisson(\lambda_1)$, $N_2 \sim Poisson(\lambda_2)$, $N_3 \sim Poisson(\lambda_3)$
Then, $N_1 + N_2 + N_3 \sim Poisson(\lambda_1 + \lambda_2 + \lambda_3)$

(3) *Changing* λ:

(3.a) 1 *unit time* $\Longrightarrow$ *m unit time*:

For example: the probability of ("in one week", happens "$\underbrace{\textit{exactly } 10 \textit{ rainstorm}}_{n=10}$") is

$$\Pr(N=10) = e^{-\lambda}\frac{\lambda^n}{n!} \quad (n=10,\ \lambda)$$

the probability of ("$\underbrace{\textit{in } 3 \textit{ week}}_{3\lambda}$", happens "$\underbrace{\textit{exactly } 10 \textit{ rainstorm}}_{n=10}$") is

$$\Pr(N=10) = e^{-(3\lambda)}\frac{(3\lambda)^n}{n!} \quad (n=10,\ 3\lambda)$$

(3.b) *car* $\Longrightarrow$ *white car*:

For example: (i) the probability of ("in one week", arrives "$\underbrace{\textit{exactly } 10 \textit{ car}}_{n=10}$")

$$\Pr(N=10) = e^{-\lambda}\frac{\lambda^n}{n!} \quad (n=10)$$

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(ii) the probability of ("in one week", arrives $\underbrace{\text{"}exactly\ 10\ white\ car\text{"}}_{n=10}$, $\underbrace{p(white) = 0.2}_{0.2\lambda}$)

$$\Pr(N = 10) = e^{-(0.2\lambda)}\frac{(0.2\lambda)^n}{n!} \quad (n = 10,\ 0.2\lambda)$$

(iii) the probability of $\underbrace{(\text{"}in\ \ 3\ week\text{"}, arrives\ \text{"}exactly\ 10\ white\ car\text{"}, p(white) =\ 0.2)}_{0.2*3*\lambda,\quad n=10}$

$$\Pr(N = 10) = e^{-(0.2*3\lambda)}\frac{(0.2*3\lambda)^n}{n!} \quad (n = 10,\ 0.2*3\lambda)$$

(iv) the probability of $\underbrace{(\text{"}in\ \ 3\ week\text{"}, arrives\ \text{"}\leq 2\ white\ car\text{"}, p(white) =\ 0.2)}_{0.2*3*\lambda,\quad n=10}$

$$\Pr(N = 0) + \Pr(N = 1) + \Pr(N = 2) \quad \text{with } 0.2*3*\lambda$$

(4) *Conditional Probability*:

Definition: Probability of "within one unit time", the "number of occurrence is $\underbrace{n}_{N=n}$", given N>1

$$\begin{aligned}
P(N &= n \mid N > 1) \\
&= \frac{P(N = n\ \cap\ N > 1)}{P(N > 1)} \\
&= \frac{P(N = n)}{P(N > 1)} \\
&= \frac{P(N = n)}{1 - P(N \leq 1)} \\
&= \frac{P(N = n)}{1 - [P(N = 0) + P(N = 1)]} \\
where\ P(N &= 0) = e^{-\lambda}\frac{\lambda^{n=0}}{(n = 0)!} \\
P(N &= 1) = e^{-\lambda}\frac{\lambda^{n=1}}{(n = 1)!} \\
P(N &= n) = e^{-\lambda}\frac{\lambda^n}{(n)!}
\end{aligned}$$