

SOA and CAS: Exam P, Probability¹

Chapter 17: Double Expectation Formulas

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(1) Conditional Variance Formula

$$\text{Var}(X) = E[\text{Var}(X | Y)] + \text{Var}[E(X | Y)] \quad (1)$$

For example (1.a): $E(X | Y) = 10y$, $\text{Var}(X | Y) = 3y^2$, $E(Y) = 5$, $\text{Var}(Y) = 12$
Question: What is $\text{Var}(X)$?

$$\begin{aligned} \text{Solve: } \text{Var}(X) &= E[\underbrace{\text{Var}(X | Y)}_{3y^2}] + \text{Var}[\underbrace{E(X | Y)}_{10y}] \\ &= E[3y^2] + \text{Var}[10y] \\ &= 3 * \underbrace{E[y^2]}_{\text{Var}[y]+[E(y)]^2} + 10^2 * \underbrace{\text{Var}[y]}_{12} \\ &= 1311 \end{aligned}$$

(2) Double Expectation Formula

$$E[g(X)] = E_Y[E_X[g(X | Y)]] \quad (2)$$

To use the “double expectation formula”, we need the following conditions:

(i) Some Y depends on / varying with X

For example: Claim loss Y depends on whether X is a minor or major accident
Claim payment Y depends on whether X is a partial or full damage

(ii) Question is about $E(Y)$

To solve the question:

Step 1, re-write $E(Y)$: $E(Y) = E(Y|X_1) * \square + E(Y|X_2) * \square + E(Y|X_3) * \square$

Step 2, use $P(X_i)$ to time each conditional expectation $E(Y | X_i)$

$$E(Y) = E(Y | X_1) * P(X_1) + E(Y | X_2) * P(X_2) + E(Y | X_3) * P(X_3)$$

For example (1.c): some claim payment Y depends on whether X is a partial or full damage
if it is a full damage, pay 15000

$$\text{Then, } E(Y) = E(Y | \text{partial}) * P(\text{partial damage}) + \underbrace{E(Y | \text{full})}_{15000} * P(\text{full damage})$$

For example (1.d): X : losses on an insurance policy]

$X \sim \text{Uniform}[0, 10000]$

Policy has a 1000 deduction

Question: Average payment made by this policy?

$$\begin{aligned} \text{Solve: } E(\text{benefit}) &= \underbrace{E(\text{benefit} | X \leq 1000)}_0 * \underbrace{P(X \leq 1000)}_{\frac{1000}{10000}} + \underbrace{E(\text{benefit} | X > 1000)}_{\text{uniform on } [1000, 10000], \text{ mean}=4500} * \underbrace{P(X > 1000)}_{\frac{9000}{10000}} \\ &= \frac{10000-1000}{2} * \frac{9000}{10000} = 4500 * 0.9 = 4050 \end{aligned}$$

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²Email: liyifinhub@outlook.com. This note was drafted when I was preparing for the exam. Please email me if you find any errors. My personal website <http://www.yilifinhub.com>