SOA and CAS: Exam P, Probability[1]
Chapter 16 and 26: Uniform Distribution

Yi Li [2]

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Chapter 16: Uniform Distribution: Single Variable
Chapter 26: Joint Uniform Distribution

(1) *Single Variable: Continuous, Conditional, Distrete, and Generalized Versions*

(1.a) *Continuous*

Definition: probability density $f(x) = \frac{1}{b-a}$ $(a \le x \le b)$

cumulative distribution $F_X(x) = \begin{cases} 0 & (x < a) \\ \frac{x-a}{b-a} & (a \le x \le b) \\ 1 & (x > b) \end{cases}$

mean and variance $E(X) = \frac{a+b}{2}$; $Var(X) = \frac{(b-a)^2}{12}$

For example: give $X \sim Uniform[0, 1000]$ with "policy limit 600"
Question: What is the "Average Benefit"?

Solve: Average benefit $E(X) = \underbrace{Expected\ Value\ over\ [0, 600]}_{part\ 1} + \underbrace{Expected\ Value\ over\ (600, 1000]}_{part\ 2}$

$$= 60\% * \frac{600-0}{2} + 40\% * 600 = 420$$

Alternative Method: $E(X) = \int_0^{600} x * \underbrace{f(x)}_{\frac{1}{1000-0}}\ dx + \int_{600}^{1000} 600 * \underbrace{f(x)}_{\frac{1}{1000-0}}\ dx = 420$

(1.b) *Conditional*

Porperty: Let $X \sim Uniform\ (a \le x \le c)$, some b > a
Then, $X|X > b \sim Uniform\ (b < x \le c)$
$X|X \le b \sim Uniform\ (a \le x \le b)$

For example:
(i) Let X be the loss amount
(ii) If the accident is *minor*, then X follows a Uniform[0, b] $\iff$ $X|minor \sim Uniform[0, b]$
(iii) If the accident is *major*, then X follows a Uniform[b, 3b] $\iff$ $X|major \sim Uniform[b, 3b]$
(v) P(a minor accident)=0.75, P(a major accident)=0.25
(vi) Median loss amount due to this accident is 672 $\iff$ $X_P^{50\%}(median\ loss) < b$
because: (1) P(a minor accident) = 0.75; (2) $X|minor \sim Uniform[0, b]$

$$\iff \underbrace{P(X < b)}_{0.75} * \underbrace{Pr(X|X < b)}_{uniform[0,\ b]} = 0.75 * \frac{672-0}{b-0} = 50\% => b = 1008$$

Then, mean(loss amount)=$\underbrace{P(X < b)}_{0.75} * \underbrace{E(X|X < b)}_{uniform[0,\ 1008]} + \underbrace{P(X \ge b)}_{0.25} * \underbrace{E(X|X \ge b)}_{uniform[1008,\ 3*1008]}$

$$=0.75 * \frac{0+1008}{2} + 0.25 * \frac{1008+3*1008}{2} = 882$$

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[2] Email: liyifinhub@outlook.com. This note was drafted when I was preparing for the exam. Please email me if you find any errors. My personal website http://www.yilifinhub.com

(1.c) *Discrete*

Definition: probability density $p(x) = \frac{1}{n}, \ \ for \ n = 1, 2, 3..., n$
mean $E(X) = \frac{1+n}{2}$
variance $Var(X) = \frac{n^2-1}{2}$ (variance formula is rarely asked in the exam)

For example: give $n = 1, 2, 3 \Longrightarrow$ the probability of each point is $\frac{1}{3}$

(1.d) *Generalized Version: Beta Distribution*

Definition: probability density $f(x, a, b) = \frac{\tau(a+b)}{\tau(a)\tau(b)} x^{a-1}(1-x)^{b-1} \ (0 \leq x \leq 1)$
where $\tau(a) = (a-1) * (a-2) * ... * 1$

When $a = b = 1$, *Beta Distribution* $\Rightarrow$ *Uniform Distribution*

Mean, Variance, and Mode of this distribution:

$$E(X) = \frac{a}{a+b} \quad Var(X) = \frac{ab}{(a+b)^2(a+b+a)} \quad mode = \frac{a-1}{a+b-2}$$

(2) *Joint Uniform Distribution: Double Variables (Continuous)*

(2.a) *Independent X and Y:* If (i) $X \sim Uniform\ [a, b]$, $Y \sim Uniform\ [c, d]$
(ii) X and Y are *independent*
Then, $f(x, y) = \frac{1}{b-a} * \frac{1}{c-d}$

(2.b) *Geometrical Method*: $Probability = \frac{A}{B}$, A : *target area*, B : *total area*

Example (2.b.1): give $x \sim Uniform$ [0, 30], $Y \sim Uniform$ [0, 20]
Question: What is $P(Y < X - 5)$?

Solve: $P(Y < X - 5) = \frac{A}{B} = \frac{trapezoidal\ area}{30*20} = \frac{\frac{1}{2}(upper\ bottom + lower\ bottom)*high}{30*20} = \frac{300}{600}$

Example (2.b.2): the joint distribution of X and Y is uniformly distributed on the circle of radius centered at the origin
Question: What is $P(X > 0.5)$?

Solve: $P(X > 0.5) = \frac{A}{B} = \frac{sector\ area\ -\ area\ of\ right\ triangle}{sector\ area} = 0.1955$

Useful problem-solving method: $\int_m^n (x+b)dx = \int_m^n (x+b)d(x+b) = \int_m^n \frac{1}{2} d(x+b)^2 = \frac{1}{2}(x+b)^2|_m^n = \frac{1}{2}(n+b)^2 - \frac{1}{2}(m+b)^2$