SOA and CAS: Exam P, Probability[1]
Chapter 15: Conditional Moments

Yi Li [2]

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(1) *Two Cases: Discrete and Continue*

(1.a) *Discrete: Property*

$$\boxed{\begin{aligned} E[g(Y) \mid X=x] &= \sum_i g(y_i) * \underbrace{P(Y \mid X=x)}_{\frac{P(Y=y_i \mid X=x)}{P_X(X=x)}},\quad \textit{where } P_X(X=x)=\sum_i p(y_i,\, x) \\ &= \sum_i g(y_i) * \frac{P(Y=y_i \mid X=x)}{\sum_i p(y_i, x)} \end{aligned}}$$

For example: give the joint density distribution function
Question: What is $Var[Y \mid X=1]$?

Joint Density		Y = 1	Y = 2	Y = 3
X	1	$\underbrace{0.1}_{(Y=1,\, X=1)}$	$\underbrace{0.2}_{(Y=2,\, X=1)}$	$\underbrace{0.3}_{(Y=3,\, X=1)}$
X	2	$\underbrace{0.1}_{(Y=1,\, X=2)}$	$\underbrace{0.2}_{(Y=2,\, X=2)}$	$\underbrace{0.1}_{(Y=3,\, X=2)}$

Solve: *Step 1:* Let $g(Y)=Y$, which can take values: 1, 2, 3

$$E[g(Y) \mid X=1] = E[Y \mid X=1] = \sum_{i=1}^{3} g(y_i) * \underbrace{P(Y \mid X=1)}_{\frac{P(Y=y_i \mid X=1)}{P_X(X=1)}}$$

$$\begin{aligned} = \quad & 1 * P(Y=1 \mid X=1) \\ & + 2 * P(Y=2 \mid X=1) \\ & + 3 * P(Y_3=3 \mid X=1) \end{aligned}$$

$$\begin{aligned} = \quad & 1 * \frac{P(Y=1,\, X=1)}{P(X=1,\, Y=1)+P(X=1,\, Y=2)+P(X=1,\, Y=3)} \\ & + 2 * \frac{P(Y=2,\, X=1)}{P(X=1,\, Y=1)+P(X=1,\, Y=2)+P(X=1,\, Y=3)} \\ & + 3 * \frac{P(Y=3,\, X=1)}{P(X=1,\, Y=1)+P(X=1,\, Y=2)+P(X=1,\, Y=3)} \end{aligned}$$

Step 2: Let $g(Y)=Y^2$, which can take values: 1, 2, 3

$$E[g(Y) \mid X=1] = E[Y^2 \mid X=1] = \sum_{i=1}^{3} g(y_i) * \underbrace{P(Y \mid X=1)}_{\frac{P(Y=y_i \mid X=1)}{P_X(X=1)}}$$

$$\begin{aligned} = \quad & 1^2 * \frac{P(Y=1,\, X=1)}{P(X=1,\, Y=1)+P(X=1,\, Y=2)+P(X=1,\, Y=3)} \\ & + 2^2 * \frac{P(Y=2,\, X=1)}{P(X=1,\, Y=1)+P(X=1,\, Y=2)+P(X=1,\, Y=3)} \\ & + 3^2 * \frac{P(Y=3,\, X=1)}{P(X=1,\, Y=1)+P(X=1,\, Y=2)+P(X=1,\, Y=3)} \end{aligned}$$

Step 3: $Var[Y \mid X=1] = E[Y^2 \mid X=1] - \{E[Y \mid X=1]\}^2$

$$\boxed{\begin{aligned} E[g(Y) \mid X = x] &= \int\limits_{over\ y} g(y) \ * \ \underbrace{f_{Y|X}(y \mid x)}_{\frac{f_{Y|X}(x,\ y)}{f_X(x)}} \ dy;\ where\ \underbrace{f_X(x)}_{Constant} = \underbrace{\int\limits_{over\ y} f(x,\ y)dy}_{Constant} \\ &= \int\limits_{over\ y} g(y) \ * \ \frac{f_{Y|X}(x,\ y)}{\underbrace{f_X(x)}_{Constant}} \ dy \\ &= \frac{1}{\underbrace{f_X(x)}_{Constant}} \int\limits_{over\ y} g(y) \ * \ f_{Y|X}(x,\ y) \ dy \end{aligned}}$$

(1.b) *Continue: Property*

For example: Joint density distribution function is

$$f(x, y) = 1.2(x^2 + y) \qquad (0 \leq x \leq 1,\ 0 \leq y \leq 1)$$

Question: What is $Var[Y \mid X = 0.2]$?

Solve: *Step 1:* Let $g(Y) = Y$

$$E[g(Y) \mid X = 0.2] = \int_0^1 y \ * \ \frac{f_{Y|X}(x{=}0.2,\ y)}{f_X(x{=}0.2)} \ dy = \int_0^1 y \ * \ \frac{1.2(0.2^2+y)}{\underbrace{\int_0^1 1.2(0.2^2 + y)dy}_{Let:\ Constant=a}} \ dy$$

$$= \frac{1}{a} \int_0^1 y \ * \ [1.2(0.2^2 + y)] \ dy$$

Step 2: Let $g(Y) = Y^2$

$$E[g(Y) \mid X = 0.2] = E[Y^2 \mid X = 0.2] = \frac{1}{a} \int_0^1 y^2 \ * \ [1.2(0.2^2 + y)] \ dy$$

Step 3: $Var[Y \mid X = 0.2] = E[Y^2 \mid X = 0.2] - \{E[Y \mid X = 0.2]\}^2$

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[2]Email: liyifinhub@outlook.com. This note was drafted when I was preparing for the exam. Please email me if you find any errors. My personal website http://www.yilifinhub.com