SOA and CAS: Exam P, Probability[1]
Chapter 14 and 29: Covariance

Yi Li [2]

January 13, 2024

Chapter 14 and 29 Covariance

(1) *Definition:*

$$\begin{aligned} Cov(X,\ Y) &= E(XY) - E(X) * E(Y) \\ &= \rho_{xy}\sigma_x\sigma_y \qquad \text{where } \rho_{xy} \text{ is } Corr(X, Y) \end{aligned}$$

Note, both $Cov(X, Y)$ and ρ_{xy} can be negative

For example: give joint distribution as below

Joint Distribution		Y		
		1	2	3
X	1	$\underbrace{0.4}_{P(X=1,Y=1)}$	$\underbrace{0.12}_{P(X=1,Y=2)}$	$\underbrace{0.08}_{P(X=1,Y=3)}$
	2	$\underbrace{0.3}_{P(X=2,Y=1)}$	$\underbrace{0.06}_{P(X=2,Y=2)}$	$\underbrace{0.04}_{P(X=2,Y=3)}$

Question: what are $Cov(X, Y)$ and $Corr(X, Y)$?

Solve: we know that

	XY					Check
Value	1	2	3	4	6	$\sum$
Probability	$\underbrace{0.4}_{P(X=1,Y=1)}$	$\underbrace{0.3+0.12=0.42}_{P(X=1,Y=2)+P(X=2,Y=1)}$	$\underbrace{0.08}_{P(X=1,Y=3)}$	$\underbrace{0.06}_{P(X=2,Y=2)}$	$\underbrace{0.04}_{P(X=2,Y=3)}$	1 *okay*

	X		Check
Value	1	2	$\sum$
Probability	$\underbrace{0.4+0.12+0.08=0.6}_{P(X=1,Y=1)+P(X=1,Y=2)+P(X=1,Y=3)}$	$\underbrace{0.3+0.06+0.04=0.4}_{P(X=2,Y=1)+P(X=2,Y=2)+P(X=2,Y=3)}$	1 *okay*

	Y			Check
Value	1	2	3	$\sum$
Probability	$\underbrace{0.4+0.3=0.7}_{P(X=1,Y=1)+P(X=2,Y=1)}$	$\underbrace{0.12+0.06=0.18}_{P(X=1,Y=2)+P(X=2,Y=2)}$	$\underbrace{0.08+0.04=0.12}_{P(X=1,Y=3)+P(X=2,Y=3)}$	1 *okay*

By definition: $Cov(X, Y) = E(XY) - E(X) * E(Y)$

where $E(XY) = 1 * 0.4 + 2 * 0.42 + 3 * 0.08 + 4 * 0.06 + 6 * 0.04 = 1.96$

$E(X) = 1 * 0.6 + 2 * 0.4 = 1.4$

$E(Y) = 1 * 0.7 + 2 * 0.18 + 3 * 0.12 = 1.42$

$E(X^2) = 1^2 * 0.6 + 2^2 * 0.4 = 2.2$

$E(Y^2) = 1^2 * 0.7 + 2^2 * 0.18 + 3^2 * 0.12 = 2.5$

[1] The purpose of the use is non-commercial research and/or private study. Please do not copy or distribute without permission of the author.

[2] Email: liyifinhub@outlook.com. This note was drafted when I was preparing for the exam. Please email me if you find any errors. My personal website http://www.yilifinhub.com

$$Var(X) = E(X^2) - [E(X)]^2 = 2.2 - 1.4^2 = 0.24$$
$$Var(Y) = E(Y^2) - [E(Y)]^2 = 2.5 - 1.42^2 = 0.4836$$
$$\sigma_X = \sqrt{Var(X)} = \sqrt{0.24} = 0.4898$$
$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{0.4836} = 0.6954$$

Thus:

$Cov(X, Y) = E(XY) - E(X) * E(Y) = 1.96 - 1.4 * 1.42 = -0.028$ *(Covariance can be negative)*

$Corr(X, Y) = \frac{Cov(X,Y)}{\sigma_x \sigma_y} = \frac{-0.028}{\sigma_x \sigma_y} = \frac{-0.028}{0.4898*0.6954} = -0.0822$ *(Correlation can be negative)*

(2) *Porperty:*

(i) $Cov(X, Y) = E(XY) - E(X) * E(Y) = \rho_{xy}\sigma_x\sigma_y$
(ii) $Cov(X,\ Y) = Cov(Y,\ X)$
(iii) $Cov(aX,\ Y) = a * Cov(X,\ Y)$
(iv) $Cov(X,\ aY + bZ) = Cov(X,\ aY) + Cov(X,\ bZ) = a * Cov(X,\ Y) + b * Cov(X,\ Z)$
(v) $Cov(aX + bY,\ Z) = Cov(aX,\ Z) + Cov(bY,\ Z) = a * Cov(X,\ Z) + b * Cov(Y,\ Z)$
(vi) $Cov(X + Y,\ X - Y) = \underbrace{Cov(X, X)}_{Var(X)} + \underbrace{Cov(X, -Y)}_{-Cov(X,Y)} + Cov(Y,\ X) + \underbrace{Cov(Y, -Y)}_{-Cov(Y,Y)= -Var(Y)} = Var(X) - Var(Y)$

(3) *Independent:*

(i) $\rho_{xy} = 0$
(ii) $Cov(X,\ Y) = Cov(Y,\ X) = \rho_{xy}\sigma_x\sigma_y = 0$ (*Note:* Independent $\Rightarrow Cov(X,\ Y) = 0$, *Not* the other way around)
(iii) $Cov(aX,\ bY) = ab * Cov(Y, X) = 0$
(iv) $Var(aX,\ bY) = a^2 * Var(X) + b^2 * Var(Y) + 2\underbrace{Cov(X, Y)}_{\rho_{xy}\sigma_x\sigma_y} = a^2Var(X) + b^2Var(Y)$

(v) $E(XY) = E(X) * E(Y)$
(vi) $E(XY^2) = E(X) * E(Y^2)$

Note, the following porperties do not require X and Y to be independent.
(a) $E(X + Y) = E(X) + E(Y)$
(b) $E(aX + bY) = aE(X) + bE(Y)$