SOA and CAS: Exam P, Probability[1]
Chapter 13 and 28: Joint Moments

Yi Li [2]

January 13, 2024

Chapter 13 Joint Moments
Chapter 28 Joint Moments for Continuous Random Variables

(1) *Two Cases: Discrete and Continue*
(1.a) *Discrete*

$$E[\ g(X,Y)\ ] = \underbrace{\sum}_{x}\ \underbrace{\sum}_{y} g(x,y)\ * P(X=x, Y=y) \tag{1}$$

For example: give the "joint probability" as follows

Joint Probability
$P(X=1,Y=1)=0.4$
$P(X=2,Y=1)=0.2$
$P(X=3,Y=1)=0.1$
$P(X=1,Y=2)=0.2$
$P(X=2,Y=2)=0.1$

Then, we have

Y	Probability
1	$P(Y=1)=P(X=1,Y=1)+P(X=2,Y=1)+P(X=2,Y=1)=0.4+0.2+0.1=0.7$
2	$P(Y=2)=P(X=1,Y=2)+P(X=2,Y=2)=0.2+0.1=0.3$

Thus, $E(Y^2) = \underbrace{(Y=1)^2}_{1^2} * P(Y=1) + \underbrace{(Y=2)^2}_{2^2} * P(Y=2) = 1^2 * 0.7 + 2^2 * 0.3$

$E(Y) = \underbrace{(Y=1)}_{1} * P(Y=1) + \underbrace{(Y=2)}_{2} * P(Y=2) = 1 * 0.7 + 2 * 0.3$

$Var(Y) = E(Y^2) - [E(Y)]^2$

(1.b) *Continue*

$$E[\ g(X,Y)\ ] = \int_{over\ y} [\int_{over\ x} g(x,y)\ * f(x,y)\ dx]\ dy \tag{2}$$

For Example (2.1): give $f(x,y) = 1.2(x^2+y)\ \ (0 \le x \le 1,\ 0 \le y \le 1)$
Question: What is $E(X+Y)=?$

Solve: we know $g(X,Y) = X+Y$ and $E[\ g(X,Y)\ ] = \int_{over\ y} [\int_{over\ x} g(x,y)\ *f(x,y)\ dx]\ dy$

$$\text{Thus, } E(X+Y) = \underbrace{\int_{\mathbf{0}}^{\mathbf{1}}}_{full\ range} [\underbrace{\int_{\mathbf{0}}^{\mathbf{1}}}_{full\ range} \underbrace{(x+y)}_{g(x,y)} * \underbrace{1.2(x^2+y)}_{f(x,y)}\ dx]\ dy = 1.2$$

---

[1] The purpose of the use is non-commercial research and/or private study. Please do not copy or distribute without permission of the author.

[2] Email: liyifinhub@outlook.com. This note was drafted when I was preparing for the exam. Please email me if you find any errors. My personal website http://www.yilifinhub.com

For Example (2.2): give $f(x, y) = 2 \;\; (0 \le x \le y \le 1)$
Question: What is $E(XY^2) = ?$

Solve: we know that $g(X, Y) = XY^2$ and $E[\ g(X, Y)\ ] = \int_{over\ y} [\int_{over\ x} g(x, y) * f(x, y)\ dx]\ dy$

$$\text{Thus, } E(XY^2) = \underbrace{\int_{\mathbf{0}}^{\mathbf{1}}}_{full\ range} [\underbrace{\int_{\mathbf{0}}^{\mathbf{y}}}_{line\ parallel\ to\ x} \underbrace{xy^2}_{g(x,y)} * \underbrace{2}_{f(x,y)}\ dx]\ dy$$

* *X range*: draw a line parallel to x-axis
* *Y range*: full range, because we have already considered all possible value of X

For Example (2.3): give $f(x, y) = 1 \;\; (0 \le x \le 1,\ 0 \le y \le 1)$
Question: What is $E(XY^2) = ?$

Solve: we know that $f(x, y) = 1 \Longrightarrow X$ and Y are independent
Thus, $E(XY^2) = E(X) * E(Y^2)$

$$\text{where } E(X) = \underbrace{\int_{\mathbf{0}}^{\mathbf{1}}}_{full\ range} x * f_X(x)\ dx; \quad (X \text{ range: } 0 \le x \le 1 \text{ as given})$$

$$f_X(x) = \int_0^1 f(x, y)\ dy = \tfrac{1}{2} \;\; (0 \le x \le 1) \quad (Y \text{ range: } 0 \le y \le 1 \text{ as given})$$

$$\text{where } E(Y^2) = \underbrace{\int_{\mathbf{0}}^{\mathbf{1}}}_{full\ range} y^2 * f_Y(y)\ dy; \quad (Y \text{ range: } 0 \le y \le 1 \text{ as given})$$

$$f_Y(y) = \int_0^1 f(x, y)\ dx = \tfrac{1}{2} \;\; (0 \le y \le 1) \; (X \text{ range: } 0 \le x \le 1 \text{ as given})$$